Perturbations of Kantowski-Sachs models with a cosmological constant

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Abstract We investigate perturbations of Kantowski-Sachs models with a positive cosmological constant, using the gauge invariant 1+3 and 1+1+2 covariant splits of spacetime together with a harmonic decomposition. The perturbations are assumed to be vorticity-free and of perfect fluid type, but otherwise include general scalar, vector and tensor modes. In this case the set of equations can be reduced to six evolution equations for six harmonic coefficients.

1 Introduction

In this work we consider perturbations of Kantowski-Sachs models with a positive cosmological constant. Some of these models can undergo an anisotropic bounce where the universe changes from a contracting to an expanding phase. A simple argument used by Börner and Ehlers, [1], to show that an isotropic bouncing universe is excluded by observations does not hold for the Kantowski-Sachs models [2]. Hence it is of interest to study the evolution and propagation of perturbations in these models and their possible effects on observables, like the Sachs-Wolfe effect [8]. To do this we use the 1+3 and 1+1+2 covariant splits of spacetime, [5, 6, 4, 3], that are suitable for perturbation theory, as they employ variables that vanish on the

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background and hence their perturbations are gauge invariant [9]. The perturbations are assumed to be vorticity-free and of perfect fluid type, but otherwise include general scalar, vector and tensor modes. The evolution equations for the perturbative variables are then derived in terms of harmonics.

2 The 1+3 and 1+1+2 covariant formalisms

A covariant formalism for the 1+3 split of spacetimes with a preferred timelike vector, u^a, was developed in [5, 6]. The projection operator onto the perpendicular 3-space is given by $h_a^b = g_a^b + u_a u^b$. With the help of this vectors and tensors can be covariantly decomposed into "spatial" and "timelike" parts. The covariant time derivative and projected spatial derivative are given by

$$\dot{\psi}_{a..b} \equiv u^c \nabla_c \psi_{a...b} \quad \text{and} \quad D_c \psi_{a...b} \equiv h_c^f h_a^d ... h_b^e \nabla_f \psi_{d...e} \tag{1}$$

respectively. The covariant derivative of the 4-velocity, u^a, can be decomposed as

$$\nabla_a u_b = -u_a A_b + D_a u_b = -u_a A_b + \frac{1}{3}\theta h_{ab} + \omega_{ab} + \sigma_{ab} \tag{2}$$

where the kinematic quantities of u^a, acceleration, expansion,vorticity and shear are defined by $A_a \equiv u^b \nabla_b u_a$, $\theta \equiv D_a u^a$, $\omega_{ab} \equiv D_{[a} u_{b]}$, and $\sigma_{ab} \equiv D_{<a} u_{b>}$ respectively. These quantities, together with the Ricci tensor (expressed via the Einstein equations by energy density μ and pressure p for a perfect fluid) and the electric, $E_{ab} \equiv C_{acbd} u^c u^d$, and magnetic, $H_{ab} \equiv \frac{1}{2} \eta_{ade} C^{de}{}_{bc} u^c$, parts of the Weyl tensor, are then used as dependent variables. From the Ricci and Bianchi identities one obtains evolution equations in the u^a direction and constraints.

A formalism for a further split (1+2) with respect to a spatial vector n^a (with $u^a n_a = 0$) was developed in [4, 3]. Projections perpendicular to n^a are made with $N_a^b = h_a^b - n_a n^b$, and in an analogous way to above "spatial" vectors and tensors may be decomposed into scalars along n^a and perpendicular two-vectors and symmetric, trace-free two-tensors as $A^a = \mathscr{A} n^a + \mathscr{A}^a$, $\omega^a = \Omega n^a + \Omega^a$, $\sigma_{ab} = \Sigma(n^a n^b - \frac{1}{2} N_{ab}) + 2\Sigma_{(a} n_{b)} + \Sigma_{ab}$ and similarly for E_{ab} and H_{ab} in terms of $\mathscr{E}$, $\mathscr{E}_a$, $\mathscr{E}_{ab}$ and $\mathscr{H}$, $\mathscr{H}_a$, $\mathscr{H}_{ab}$ respectively. Derivatives along and perpendicular to n^a are

$$\hat{\psi}_{a...b} \equiv n^c D_c \psi_{a...b} = n^c h_c^f h_a^d ... h_b^e \nabla_f \psi_{d...e} \text{ and } \delta_c \psi_{a...b} \equiv N_c^f N_a^d ... N_b^e D_f \psi_{d...e} \tag{3}$$

respectively. Similarly to the decomposition of $\nabla_a u_b$, $D_a n_b$ and $\dot{n}_a$ can be decomposed into further 'kinematical' quantities of n^a as

$$D_a n_b = n_a a_b + \frac{1}{2}\phi N_{ab} + \xi \varepsilon_{ab} + \zeta_{ab} \quad \text{and} \quad \dot{n}_a = \mathscr{A} u_a + \alpha_a \tag{4}$$

where $a_a \equiv \hat{n}_a$, $\phi \equiv \delta_a n^a$, $\xi \equiv \frac{1}{2} \varepsilon^{cabd} \delta_a n_b u_c n_d$, $\zeta_{ab} \equiv \delta_{\{a} n_{b\}}$, $\mathscr{A} \equiv n^a A_a$, $\alpha_a \equiv N_a^b \dot{n}_b$.

The Ricci and Bianchi identities are then written as evolution and propagation equations in the u^a and n^a directions and constraints.

3 Perturbations of Kantowski-Sachs

As backgrounds we take the Locally Rotationally Symmetric (LRS) Kantowski-Sachs cosmologies [7]

$$ds^2 = -dt^2 + a_1^2(t)dz^2 + a_2^2(t)\left(d\vartheta^2 + \sin^2\theta d\varphi^2\right) \tag{5}$$

with cosmological constant $\Lambda > 0$ and matter given by a perfect fluid with barytropic equation $p = p(\mu)$. The shear Σ, energy density μ and the expansion θ evolve as

$$\dot{\Sigma} = -\frac{1}{2}\Sigma^2 - \frac{2}{3}\Sigma\theta - \mathscr{E},\ \ \dot{\mu} = -\theta(\mu + p),\ \ \dot{\theta} = (\Lambda - \frac{1}{2}\mu - \frac{3}{2}p) - \frac{1}{3}\theta^2 - \frac{3}{2}\Sigma^2 \tag{6}$$

where the electric part of the Weyl tensor is $\mathscr{E} = -\frac{2}{3}\mu - \frac{2}{3}\Lambda - \Sigma^2 + \frac{2}{9}\theta^2 + \frac{1}{3}\Sigma\theta$.

Instead of the background variables $\theta, \Sigma, \mathscr{E}, \mu$ we use their gradients

$$W_a \equiv \delta_a\theta, \quad V_a \equiv \delta_a\Sigma, \quad X_a \equiv \delta_a\mathscr{E}, \quad \mu_a \equiv \delta_a\mu\,, \tag{7}$$

which vanish on the background and hence are gauge invariant (the derivatives $\hat{\theta} \equiv n^a D_a\theta$ etc. can be given in terms of the δ_a derivatives due to commutation relations in the case of no vorticity). Similar variables vanishing on the background are $a_a, \phi, \xi, \zeta_{ab}, \alpha_a, \mathscr{A}, \mathscr{A}_a, \Sigma_a, \Sigma_{ab}, \mathscr{E}_a, \mathscr{E}_{ab}, \mathscr{H}, \mathscr{H}_a, \mathscr{H}_{ab}$ where a_a can be put to zero by choice of frame.

The scalar, vector and tensor variables are expanded in harmonics according to

$$\begin{aligned}\Psi &= \sum_{k_\parallel, k_\perp} \Psi_{k_\parallel k_\perp} P_{k_\parallel} Q_{k_\perp}\,, \quad \Psi_a = \sum_{k_\parallel, k_\perp} P_{k_\parallel}\left(\Psi^V_{k_\parallel k_\perp} Q^{k_\perp}_a + \overline{\Psi}^V_{k_\parallel k_\perp}\overline{Q}^{k_\perp}_a\right),\\ \Psi_{ab} &= \sum_{k_\parallel, k_\perp} P_{k_\parallel}\ \left(\Psi^T_{k_\parallel, k_\perp} Q^{k_\perp}_{ab} + \overline{\Psi}^T_{k_\parallel, k_\perp}\overline{Q}^{k_\perp}_{ab}\right)\end{aligned} \tag{8}$$

where $Q_{k_\perp}$, $Q^{k_\perp}_a$, $\overline{Q}^{k_\perp}_a$, $Q^{k_\perp}_{ab}$ and $\overline{Q}^{k_\perp}_{ab}$ are harmonics on the 2-spheres of constant z and $P_{k_\parallel}$ the corresponding expansion functions in the z-direction.

All coefficients can be given in terms of $\mu^V_{k_\parallel, k_\perp}$, $\Sigma^T_{k_\parallel, k_\perp}$, $\mathscr{E}^T_{k_\parallel, k_\perp}$, $\overline{\mathscr{H}}^T_{k_\parallel, k_\perp}$ and $\overline{\mathscr{E}}^T_{k_\parallel, k_\perp}$, $\mathscr{H}^T_{k_\parallel, k_\perp}$, so the system has six degrees of freedom. The first four coefficients form a closed system of evolution equations coupled to the density gradient, in agreement with the results for scalar perturbations in [2]. This reads

$$\dot{\mu}^V_{k_\parallel, k_\perp} = \left[\frac{\Sigma}{2}\left(1 - 6\frac{\mu + p}{B}\right) - \frac{4\theta}{3}\right]\mu^V_{k_\parallel, k_\perp} +$$

$$\frac{a_2}{2}(\mu+p)\left[(1-C)\left(B\Sigma^T_{k_\parallel,k_\perp}+\mathscr{E}^T_{k_\parallel,k_\perp}\right)-P\overline{\mathscr{H}}^T_{k_\parallel,k_\perp}\right],$$

$$\dot{\Sigma}^T_{k_\parallel,k_\perp}=-\frac{1}{a_2(\mu+p)}\frac{dp}{d\mu}\mu^V_{k_\parallel,k_\perp}+\left(\Sigma-\frac{2\theta}{3}\right)\Sigma^T_{k_\parallel,k_\perp}-\mathscr{E}^T_{k_\parallel,k_\perp}\,,$$

$$\dot{\mathscr{E}}^T_{k_\parallel,k_\perp}=-\frac{3\Sigma}{2a_2B}\mu^V_{k_\parallel,k_\perp}-\frac{\mu+p}{2}\Sigma^T_{k_\parallel,k_\perp}-\frac{3}{2}(F+\Sigma C)\mathscr{E}^T_{k_\parallel,k_\perp}+\frac{P}{2}\overline{\mathscr{H}}^T_{k_\parallel,k_\perp}\,,$$

$$\dot{\overline{\mathscr{H}}}^T_{k_\parallel,k_\perp}=-\frac{ik_\parallel}{a_1a_2B}\mu^V_{k_\parallel,k_\perp}-R\overline{\mathscr{H}}^T_{k_\parallel,k_\perp}-\frac{ik_\parallel}{a_1}\left[1-\frac{3}{2}\left(C-\frac{\mathscr{E}}{B}\right)\right]\mathscr{E}^T_{k_\parallel,k_\perp}\,, \tag{9}$$

where we have introduced the notations $B=\frac{2k_\parallel^2}{a_1^2}+\frac{k_\perp^2}{a_2^2}+\frac{9}{2}\Sigma^2+3\mathscr{E}$, $\quad C=B^{-1}\left(\frac{2-k_\perp^2}{a_2^2}+3\mathscr{E}\right)$, $D=C+\frac{\mu+p}{B}$, $E=\frac{\Sigma}{2}\left(C-\frac{\mathscr{E}}{B}\right)+\frac{\theta\mathscr{E}}{3B}$, $F=\Sigma+\frac{2\theta}{3}$, $P=\frac{a_1}{ik_\parallel}\left[\frac{2k_\parallel^2}{a_1^2}(1-C)-\frac{k_\perp^2}{a_2^2}\frac{2-k_\perp^2}{a_2^2B}\right]$ and $R=\frac{3}{2}F-\left(\Sigma+\frac{\theta}{3}\right)\frac{k_\perp^2}{a_2^2B}-\frac{1}{2B}\left(\Sigma-\frac{2\theta}{3}\right)\left(D-\frac{2k_\parallel^2}{a_1^2}\right)$. The two last coefficents form a closed system for free waves

$$\dot{\overline{\mathscr{E}}}^T_{k_\parallel,k_\perp}=-\frac{3}{2}(F+\Sigma D)\overline{\mathscr{E}}^T_{k_\parallel,k_\perp}+\frac{ik_\parallel}{a_1}(1-D)\mathscr{H}^T_{k_\parallel,k_\perp},$$

$$\dot{\mathscr{H}}^T_{k_\parallel,k_\perp}=-\frac{a_1}{2ik_\parallel}\left(\frac{2k_\parallel^2}{a_1^2}-BC+9\Sigma E\right)\overline{\mathscr{E}}^T_{k_\parallel,k_\perp}-\frac{3}{2}(2E+F)\mathscr{H}^T_{k_\parallel,k_\perp}\,. \tag{10}$$

These sets of equations can be used to study the propagation of gravitational waves and the coupling between scalar and tensor perturbations. Furthermore, from the null geodesics of photons, equations for the redshift in different directions can be given completely in terms of the 1+1+2 quantities. From their solutions the Sachs-Wolfe effect and the corresponding variations in the CMB temperature can be calculated.

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References

1. G. Börner and J. Ehlers, Astron. Astrophys., **204**, 1 (1988)
2. M. Bradley, P. K. S. Dunsby, M. Forsberg, and Z. Keresztes, Class. Quantum Grav. **29**, 095023 (2012).
3. C.A. Clarkson, Phys. Rev. D, **76**, 104034 (2007).
4. C. A. Clarkson and R. Barrett, Class. Quantum Grav. **20**, 3855 (2003).
5. G.F.R Ellis and M. Bruni, Phys. Rev. D, **40**, 1804 (1989).
6. G.F.R Ellis and H. van Elst, in M. Lachieze-Rey (ed.), *Theoretical and Observational Cosmology*, NATO Science Series, Kluwer Academic Publishers (1998), arXiv:gr-qc/9812046v5.
7. R. Kantowski and R.K. Sachs, J. Math. Phys. **7**, 443 (1967).
8. R. K. Sachs, and A. M. Wolfe, Astrophys. J. **147**, 73 (1967)
9. J.M. Stewart and M. Walker, Proc. R. Soc. London, **A341**, 49 (1974).